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On Parallel Displacement Within the Light Cone and Its Application in the Electrodynamics of Charges Moving with the Velocity of Light

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Abstract

It is shown that there exists on the light cone an affine connection which is metric, semisymmetric and locally integrable. There exists a correspondence between this connection and a system of charges moving with the velocity of light. The correspondence reveals in the case of commensurable charges a symmetry which disappears for non-commensurable charges.

1. *Introduction*

The line element of the Minkowski space referred to the stereographic coordinates has the form

$$-ds^2 = dt^2 - dr^2 - r^2\left(1 + \frac{x^2+y^2}{4}\right)^{-2}(dx^2 + dy^2) \tag{1.1}$$

On the hypersurface $t = r$

$$ds^2 = r^2\left(1 + \frac{x^2+y^2}{4}\right)^{-2}(dx^2 + dy^2) = g_{ik}\,dx^i\,dx^k \tag{1.2}$$

where $x^1 = r$, $x^2 = x$, $x^3 = y$. Since $\det(g_{ik}) = 0$, there is no natural notion of parallelism within the light cone. Our aim is to introduce an affine connection within the light cone and to describe some of its properties.

An affine connection Γ^i_{kl} is called metric if

$$\nabla_i g_{kl} \overset{\mathrm{df}}{=} \partial_i g_{kl} - \Gamma^s_{ik} g_{sl} - \Gamma^s_{il} g_{ks} = 0 \tag{1.3}$$

it is called symmetric if $\Gamma^i_{kl} - \Gamma^i_{lk} = 0$ and semisymmetric if there exists a vector S_i such that $\Gamma^i_{kl} - \Gamma^i_{lk} = S_k \delta_l{}^i - S_l \delta_k{}^i$. One can prove the theorem (Vogel, 1965): if rank of the matrix (g_{ik}), $i, k = 1, 2, \ldots, n$, is $n-1$, then there exists a metric and symmetric connection Γ^i_{kl} if and only if there exists locally a coordinate system in which $g_{1i}=0$, $\partial_1 g_{ik}=0$. Since the second condition is not satisfied for the light cone, we have to make a choice: either to assume that there is no torsion but a vector, displaced in a parallel manner, changes its length, or to accept the fact that the light cone is a space naturally endowed with torsion. The first possibility is chosen by Lemmer (1965) and Daŭtcourt (1967); we prefer the second one, for the following reason. Lemmer constructs a parallel displacement by means of the invariant rigging of a null hypersurface introduced by Jordan, Ehlers and Sachs. The rigging is uniquely determined by a family of null hypersurfaces. However, for a single light cone in the Minkowski space no such natural rigging exists; it is impossible to define any rigging in a Lorentz invariant way.

We shall adopt the following, manifestly Lorentz invariant procedure: since equations (1.3) do not determine the unknown functions Γ^i_{kl} uniquely, we add further conditions:

(a) the connection should preserve the Lorentz invariant volume μ of the light cone:

$$\nabla_i \mu \stackrel{\text{df}}{=} \partial_i \mu - \Gamma^s_{is}\mu = 0 \tag{1.4}$$

in the stereographic coordinates μ is equal to rf^2 where

$$\frac{1}{f} = 1 + \tfrac{1}{4}(x^2+y^2) \tag{1.5}$$

(b) the connection should be semisymmetric;

(c) it should be locally integrable, i.e. there should exist three linearly independent vector fields $\underset{s}{w}_i$, $s=1,2,3$, such that

$$\nabla_i \underset{s}{w}_k \stackrel{\text{df}}{=} \partial_i \underset{s}{w}_k - \Gamma^l_{ik}\underset{s}{w}_l = 0 \tag{1.6}$$

It turns out that all the conditions can be simultaneously satisfied. Omitting simple calculations, we give the result written in the stereographic coordinates. The vector S_i resulting from condition (b) is a gradient

$$S_i = \partial_i G \tag{1.7}$$

where

$$G(r,x,y) = \ln(rf) + G_0(x,y) \tag{1.8}$$

and $G_0(x,y)$ is an arbitrary harmonic function: $\partial_{22}G_0 + \partial_{33}G_0 = 0$. The parallel vectors $\underset{s}{w}_i$ have the form

$$\underset{1}{w}_i = S_i,\ \underset{2}{w}_i = (0, rf\cos F, rf\sin F),\ \underset{3}{w}_i = (0, -rf\sin F, rf\cos F) \tag{1.9}$$

where F is a harmonic function connected with G_0 by the Cauchy–Riemann conditions: $\partial_2 F = -\partial_3 G_0$, $\partial_3 F = \partial_2 G_0$. The connection Γ^i_{kl} may be obtained algebraically as the solution of equation (1.6); it is therefore the Weitzenböck connection determined by parallel vectors $\underset{s}{w}_i$.

2. *The Topological Structure of the Light Cone and Global Integrability Conditions*

There exists in nature a simple realisation of the light cone: it is the set of all energy-momentum states of the photon. It is not necessary to think that this set is immersed in a four-dimensional space; we can and, in fact, we should treat it as a three-dimensional manifold which exists independently of its relation to the four-dimensional space. But there is no photon with frequency equal exactly to zero. This means geometrically that the light cone is a three-dimensional space with a hole. This fact may be understood as an affine property of the light cone; given affine connection we can construct all possible geodesic lines. It follows then from our formulae that the affine distance from any point of the light cone to the origin is infinite. Consequently the origin of the light cone does not belong to it.

Suppose now that a vector is displaced in a parallel manner along a closed curve; the vector will not return back to its original direction unless the change of F along any closed curve is equal to $2n\pi$, $n = 0, \pm1, \pm2, \ldots$. This may be interpreted intuitively as follows. The affine connection is determined by a harmonic function $G_0(x, y)$, which we may interpret as a plane electrostatic potential generated by a distribution of charges, dipoles, quadrupoles, etc. The global integrability condition $\Delta F = 2n\pi$ means that charges generating G_0 must be integers. Of course, these charges have no physical meaning, they are just singularities of G_0; we shall see, however, that there is some connection between these charges and the real ones.

3. *Lorentz Invariance of the Affine Connection*

When we say that a geometric object is invariant with respect to the Lorentz group, we usually mean that all six Lie derivatives of this object vanish. The metric (1.2) is invariant in this sense. It turns out, however, that no metric affine connection can be Lorentz invariant. This fact may be astonishing since we have introduced the affine connection by a manifestly Lorentz invariant procedure. To clarify the problem let us consider transformation properties of $G_0(x, y) = G_0(z)$, $z = x + iy$. Since $G(r, z) = \ln(rf(z)) + G_0(z)$ is a scalar

$$G(r', z') = G(r(r', z'), z(z')) = \ln(r'f(z')) + G_0'(z') \tag{3.1}$$

The Lorentz transformation is a homographic substitution

$$z = \frac{\alpha z' + \beta}{\gamma z' + \delta}, \qquad \alpha\delta - \beta\gamma = 1 \tag{3.2}$$

Transformation of r may be found if one takes into account that the Lorentz transformation is a rigid motion of the light cone. We find from (3.1) and (3.2)

$$G_0'(z') = G_0(z(z')) - \ln\left|\frac{dz}{dz'}\right| \tag{3.3}$$

The additional term in (3.3) is the electrostatic potential of an isolated charge; consequently, if $G_0(z)$ is a harmonic function, $G_0'(z')$ is also a harmonic function but, in general, different from $G_0(z)$. This explains why the Lie derivatives of the connection cannot simultaneously vanish.

We conclude that the affine connection is not an invariant but generates a representation of the Lorentz group: the formulae (1.8) and (1.9) have the same form in all Lorentz systems, but the harmonic function G_0, when transformed, becomes a different harmonic function.

It is clear that $G_0'(z')$ will be a function of the same type as $G_0(z)$ if $G_0(z)$ is the electrostatic potential generated by point charges:

$$G_0(z) = \sum_n q_n \ln|z - z_n| \tag{3.4}$$

where q_n are integers. We have

$$\begin{aligned} G_0'(z') &= \sum_n q_n \ln\left|\frac{\alpha z' + \beta}{\gamma z' + \delta} - z_n\right| - \ln\frac{1}{|\gamma z' + \delta|^2} \\ &= \left(2 - \sum_n q_n\right) \ln|\gamma z' + \delta| + \sum_n q_n \ln|z' - z_n'| + \sum_n q_n \ln|\alpha - \gamma z_n| \end{aligned} \tag{3.5}$$

The last term in (3.5) is a constant which does not affect the affine connection. We see, therefore, that the function $G_0'(z')$ will be of the same type as $G_0(z)$ if

$$\sum_n q_n = 2 \tag{3.6}$$

4. *The Electric Current of a System of Point Charges and Its Connection with the Affine Properties of the Light Cone*

Let $j_\mu(k)$, $\mu = 0, 1, 2, 3$ be the Fourier transform of the electric current of a system of point charges. It is usual to divide $j_\mu(k)$ on two parts: the part regular for $k_0 \to 0$ and the singular part, which generates the so-called infrared radiation. The singular part of the current equals (Jauch & Rohrlich, 1955)

$$j_\mu(k) = \sum_s e_s \left\{\frac{\underset{s}{u}_\mu(+\infty)}{k_\lambda \underset{s}{u}^\lambda(+\infty)} - \frac{\underset{s}{u}_\mu(-\infty)}{k_\lambda \underset{s}{u}^\lambda(-\infty)}\right\} \tag{4.1}$$

here e_s are charges and $\underset{s}{u}_\mu(\pm\infty)$ velocities at plus and minus infinity. It is assumed in equation (4.1) that each particle preserves its identity in the process of scattering; it is not necessary to make so stringent an assumption. We may imagine that particles disintegrate in the process of scattering. Moreover, we make the following convention: contributions to the asymptotic current are written with their proper sign if they come from plus infinity and with changed sign if they come from minus infinity. With this convention the current may be written in a more symmetric form

$$j_\mu(k) = \sum_s e_s \frac{\underset{s}{u}_\mu}{k_\lambda \underset{s}{u}^\lambda}, \qquad \sum_s e_s = 0 \tag{4.2}$$

The number of terms in (4.2) can be odd; for example, if a particle disintegrates on two particles, there are three terms in $j_\mu(k)$.

The current (4.2) is conserved: $k^\mu j_\mu(k) = 0$; this means that the current lies upon the cone $k^\mu k_\mu = 0$. Let $k_\mu = k_\mu(x^i)$ be the parametrisation of the cone by means of the stereographic coordinates; since the current is an internal vector, we may refer it to the coordinate system x^i:

$$j_i(x) = j_\mu(k)\frac{\partial k^\mu}{\partial x^i} = \partial_i \sum_s e_s \ln(k_\lambda \underset{s}{u}^\lambda) \tag{4.3}$$

The current (4.3) takes on a particularly simple form if one assumes that all charges move with the velocity of light. Such an assumption might be unphysical; if, however, we treat charges as given sources of the Maxwell field, there is mathematically nothing wrong about charges moving with the velocity of light (Bonnor, 1969).

For light-like velocities

$$j_i = \partial_i \sum_s e_s \ln |z - z_s|^2 \tag{4.4}$$

where z_s is a complex number corresponding to the null direction $\underset{s}{u}^\mu$ according to the general rule

$$\tfrac{1}{2}z = \frac{u^1 + iu^2}{u^0 + u^3} \tag{4.5}$$

For every current (4.4) there exists one and only one affine connection such that

$$\nabla_i j_k + S_i j_k = 0 \tag{4.6}$$

The condition (4.6) means geometrically that the current is proportional to a parallel vector. The connection is generated by the function

$$G_0(z) = -\ln \left|\frac{d\varphi}{dz}\right| \tag{4.7}$$

where

$$\varphi(z) = \sum_s e_s \ln (z - z_s)^2 \tag{4.8}$$

5. *Partitions of a Charge Moving with the Velocity of Light*

Let us assume that the charge $-e_1$ disintegrates on two charges e_2 and e_3; of course $e_1 + e_2 + e_3 = 0$. In this case

$$\varphi(z) = 2 \ln (z - z_1)^{e_1}(z - z_2)^{e_2}(z - z_3)^{e_3} \tag{5.1}$$

and

$$G_0(z) = -\ln \left|\frac{d\varphi}{dz}\right| = \ln \left|\frac{(z - z_1)(z - z_2)(z - z_3)}{z - z_4}\right| + \text{const} \tag{5.2}$$

where

$$z_4 = \frac{e_1 z_2 z_3 + e_2 z_3 z_1 + e_3 z_1 z_2}{e_1 z_1 + e_2 z_2 + e_3 z_3} \tag{5.3}$$

The current is determined by three null directions z_1, z_2, z_3 and by three charges e_1, e_2, e_3. Since these charges satisfy the conservation law and since an overall factor in j_i may be arbitrarily fixed by an appropriate choice of the unit charge, there is effectively one number which determines the current, for example the ratio e_1/e_2. On the other hand $G_0(z)$ is determined by four null directions z_1, z_2, z_3 and z_4. This means that we have received a sort of space-time interpretation of a ratio of two charges. In particular, if $G_0(z)$ is given and we know or assume that it corresponds to the process of disintegration of one charge on two charges, we can determine the ratio e_1/e_2 by means of the harmonic ratio

$$(z_1 z_2 z_3 z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{5.4}$$

which is a space-time invariant.

We know that all charges in nature happen to be commensurable. What is the space-time interpretation of this fact? The answer is very simple.

Let K be the circle through z_1, z_2, z_3 and let K_{12} be the circle through z_1 and z_2 orthogonal to K; similarly let K_{23} (K_{31}) be the circle through z_2 and z_3 (z_3 and z_1) orthogonal to K. Let us reflect z_1 in K_{23}, z_2 in K_{31} and z_3 in K_{12}; the images z_1', z_2' and z_3' will fall on K.

A simple calculation shows that each image, if taken as z_4, determines a partition of charge $e_1:e_2:e_3 = 1:1:-2$ or $1:-2:1$ or $-2:1:1$. One can continue this procedure, reflecting the images again; one obtains six new points which, if taken as z_4, determine partitions $e_1:e_2:e_3 = 1:2:-3$, $1:-3:2$, etc. This Lorentz invariant procedure may be further continued: every reflexion gives a point which, if taken as z_4, determines a partition $e_1:e_2:e_3 = n_1:n_2:n_3$, $n_1 + n_2 + n_3 = 0$, where n_1, n_2 and n_3 are integers

Our results may be summed up as follows: the affine connection within the light cone, generated in the described way by the current j_μ, possesses a higher degree of space-time symmetry if the charges are commensurable. It is difficult to say if this has any physical significance since our results can be proved only for charges moving with the velocity of light. It should be remembered, however, that at present there is no understandable physical principle which would explain commensurability of charges; therefore it might be interesting that for charges moving with the velocity of light such a principle can be formulated.

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